

Title	ζ_p -th power relations and Euler-Carlitz relations among multizeta values (Algebraic Number Theory and Related Topics 2013)
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Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2015), B53: 13-29
Issue Date	2015-09
URL	http://hdl.handle.net/2433/241272
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Type	Departmental Bulletin Paper
Textversion	publisher

p-th power relations and Euler-Carlitz relations among multizeta values

By

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Abstract

In this paper, we study p -th power relations and Euler-Carlitz relations among multizeta values in characteristic p . By definition, two multizeta values have the p -th power relation if their indices map to each other multiplying by some power of p . The multizeta values of depth one at “even” integers satisfy Euler-Carlitz relations which are analogues of the relations among the Riemann zeta values at positive even integers. We prove that all algebraic relations among given multizeta values come from p -th power relations and Euler-Carlitz relations if their indices satisfy some conditions.

§ 1. Introduction

Let $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a d -tuple ($d \geq 1$) of positive integers such that $n_1 \geq 2$. The sum

$$\zeta_{\mathbb{Z}}(\underline{n}) := \sum_{m_1 > \dots > m_d > 0} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{R}$$

is called the *multiple zeta value* (MZV) and studied by many mathematicians. Many relations over \mathbb{Q} among MZV are known. For example, Euler showed that

$$\zeta_{\mathbb{Z}}(n) \in (2\pi\sqrt{-1})^n \cdot \mathbb{Q}^{\times}$$

for each positive even integer $n \geq 2$. We also have the *harmonic product formula*. The simplest case is as follows:

$$\zeta_{\mathbb{Z}}(n_1)\zeta_{\mathbb{Z}}(n_2) = \zeta_{\mathbb{Z}}(n_1, n_2) + \zeta_{\mathbb{Z}}(n_2, n_1) + \zeta_{\mathbb{Z}}(n_1 + n_2).$$

Received March 31, 2014. Revised January 23, 2015.

2010 Mathematics Subject Classification(s): 11J93 (Primary) 11M38, 11G09 (Secondary).

Key Words: positive characteristic multizeta values, algebraic independence, t -motives.

Partially supported by the JSPS Research Fellowships for Young Scientists.

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We also want to know the linear/algebraic independence among given MZVs. However, we do not even know whether $\zeta_{\mathbb{Z}}(n)$ is transcendental over \mathbb{Q} for each positive odd integer $n \geq 3$. In general, such problems seem very difficult.

Next, we consider the positive characteristic case. We fix a prime number p and its power q . Let θ be a variable, $A := \mathbb{F}_q[\theta]$ the one variable polynomial ring over \mathbb{F}_q , $K := \mathbb{F}_q(\theta)$ the fraction field of A , $K_{\infty} := \mathbb{F}_q((\theta^{-1}))$ the ∞ -adic completion of K , \mathbb{C}_{∞} the ∞ -adic completion of an algebraic closure of K_{∞} , and \overline{K} the algebraic closure of K in \mathbb{C}_{∞} . Let $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a d -tuple ($d \geq 1$) of positive integers. Such an \underline{n} is called an *index of weight* $\text{wt}(\underline{n}) := \sum n_i$ and *depth* $\text{dep}(\underline{n}) := d$. For an index \underline{n} , Thakur ([9, Section 5.10]) defined the *multizeta value* in characteristic p by

$$\zeta(\underline{n}) := \sum_{\substack{a_1, \dots, a_d \in A: \text{monic} \\ \deg(a_1) > \dots > \deg(a_d) \geq 0}} \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in K_{\infty}.$$

We are also interested in determining all relations over \overline{K} among given MZVs. For an index $\underline{n} = (n_1, \dots, n_d)$ and an integer $e \in \mathbb{Z}$, we set

$$p^e \underline{n} := (p^e n_1, \dots, p^e n_d) \in \mathbb{Z}[1/p]^d.$$

If $p^e \underline{n} \in \mathbb{Z}^d$, the p -th power relation

$$\zeta(p^e \underline{n}) = \zeta(\underline{n})^{p^e}$$

follows immediately from the definition of $\zeta(\underline{n})$. The MZVs of depth one are defined by Carlitz ([4]) and called the *Carlitz zeta values*. Carlitz showed the relation

$$\zeta(n) = \tilde{\pi}^n \cdot \frac{B_n}{\Gamma_{n+1}}$$

for each positive integer $n \geq 1$ which is divisible by $q - 1$, where

$$\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} \cdot K_{\infty}^{\times}$$

is the *Carlitz period*, $B_n \in A$ is the *Bernoulli-Carlitz number* and $\Gamma_{n+1} \in A$ is the *factorial* of Carlitz (see Section 2). These relations are called the *Euler-Carlitz relations*. These are analogues of Euler's relations of the special zeta values at positive even integers. We say that a positive integer $n \geq 1$ is “even” (resp. “odd”) if n is divisible (resp. not divisible) by $q - 1$. After works of Wade ([10]) and Yu ([11], [12]), finally Chang and Yu ([5, Corollary 4.6]) proved that all relations over \overline{K} among the Carlitz zeta values come from p -th power relations and Euler-Carlitz relations. This means that if $n_1, \dots, n_d \geq 1$ are positive “odd” integers such that n_i/n_j is not an integral power of

p for each $i \neq j$, then $\tilde{\pi}, \zeta(n_1), \dots, \zeta(n_d)$ are algebraically independent over \overline{K} . This is generalized in [7, Theorem 1.1] as follows: if n_1, \dots, n_d satisfy the above assumptions, then the set

$$\{\tilde{\pi}\} \cup \{\zeta(\underline{m}) | \underline{m} \in \text{Sub}(\underline{n})\}$$

has $1 + d(d+1)/2$ elements and these elements are algebraically independent over \overline{K} , where for an index $\underline{n} = (n_1, \dots, n_d)$, we set

$$\text{Sub}(\underline{n}) := \{(n_j, n_{j+1}, \dots, n_i) | 1 \leq j \leq i \leq d\}.$$

Our results in this paper contain this as a special case.

To explain the results, let us introduce some notations.

Definition 1.1. Let $\underline{n} = (n_1, \dots, n_d)$ be an index.

(1) We set

$$\text{Sub}'(\underline{n}) := \{(n_{i_1}, \dots, n_{i_r}) | 1 \leq r \leq d, 1 \leq i_1 < \dots < i_r \leq d\}.$$

Thus we have

$$\text{Sub}'(\underline{n}) \supset \text{Sub}(\underline{n}), \quad \#\text{Sub}'(\underline{n}) \leq 2^d - 1 \quad \text{and} \quad \#\text{Sub}(\underline{n}) \leq \frac{d(d+1)}{2}.$$

(2) For each $1 \leq j < i \leq d+1$, we set

$$\underline{n}_{ij} := (n_j, n_{j+1}, \dots, n_{i-1}).$$

Thus we have

$$\text{Sub}(\underline{n}) = \{\underline{n}_{ij} | 1 \leq j < i \leq d+1\}.$$

(3) Let \underline{n}' be another index. We say that \underline{n} and \underline{n}' are equivalent and denote by $\underline{n} \sim \underline{n}'$ if there exists an integer $e \in \mathbb{Z}$ such that $\underline{n} = p^e \underline{n}'$, or both $\underline{n}, \underline{n}'$ are of depth 1 and $\underline{n} = (m), \underline{n}' = (m')$ for some $m, m' \in (q-1)\mathbb{Z}$ (hence $\text{dep}(\underline{n}) = \text{dep}(\underline{n}')$ if $\underline{n} \sim \underline{n}'$). For positive integers m and m' , we write $m \sim m'$ if the indices (m) and (m') of depth one are equivalent.

(4) Let S be a set of indices. We denote by S/\sim the quotient set of S by the equivalence relation \sim .

The purpose of this paper is to generalize the result [7, Theorem 1.1] to the following three directions:

- n_i/n_j may be an integral power of p ,
- n_i may be “even”,
- treat elements $\zeta(\underline{m})$ for $\underline{m} \in \text{Sub}'(\underline{n})$.

We expect that if the given indices satisfy some “good” conditions, then all algebraic relations over \overline{K} among the multizeta values at such points come from p -th power relations and Euler-Carlitz relations. In fact, in this paper, we prove the following theorems:

Theorem 1.2. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index such that the n_i ’s are “odd” and distinct from each other. Assume that there exists exactly one pair $j_1 < j_2$ such that $n_{j_1} \sim n_{j_2}$. We set*

$$S := \{\underline{m} \mid \underline{m} \in \text{Sub}'(\underline{n}), (n_{j_1}, n_{j_2}) \notin \text{Sub}(\underline{m})\}.$$

Then we have

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{m}) \mid \underline{m} \in S) = \#((\{(q-1)\} \cup S) / \sim).$$

Note that the condition $(n_{j_1}, n_{j_2}) \notin \text{Sub}(\underline{m})$ means that \underline{m} is not an index of the form $\underline{m} = (\dots, n_{j_1}, n_{j_2}, \dots)$.

As a consequence of Theorem 1.2, we have the following corollary:

Corollary 1.3. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index of positive “odd” integers such that n_i/n_j is not an integral power of p for each $i \neq j$. Then we have*

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{m}) \mid \underline{m} \in \text{Sub}'(\underline{n})) = 2^d.$$

We also have the following theorem:

Theorem 1.4. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index such that the n_i ’s are “odd” and distinct from each other. Assume that there exists exactly one pair $j_1 \neq j_2$ such that $n_{j_1} \sim n_{j_2}$. Then we have*

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{m}) \mid \underline{m} \in \text{Sub}(\underline{n})) = \#((\{(q-1)\} \cup \text{Sub}(\underline{n})) / \sim) = \frac{d(d+1)}{2}.$$

Remark. We do not know in general that when

$$\tilde{\pi}, \zeta(n) \text{ and } \zeta((q-1)m, n) \quad (\text{or } \tilde{\pi}, \zeta(n) \text{ and } \zeta(n, (q-1)m))$$

are algebraically independent over \overline{K} , where n is “odd”. Thus we do not treat “even” integers in Theorems 1.2 and 1.4. When we treat “even” integers, we need to assume that the above elements are already algebraically independent over \overline{K} as in Theorem 1.5.

For a set S of indices, we define a set $[S]$ by

$$[S] := \{\underline{m} : \text{index} \mid \underline{m} \sim \underline{n} \text{ for some } \underline{n} \in S\}.$$

Theorems 1.2 and 1.4 follow from the following theorem:

Theorem 1.5. *Let $\underline{n}^{(m)} = (n_1^{(m)}, \dots, n_{d_m}^{(m)})$ ($1 \leq m \leq k$, $k \geq 2$) be indices. If the following conditions (1) \sim (5) hold, then we have*

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{n}) | \underline{n} \in \cup_{m=1}^k \text{Sub}(\underline{n}^{(m)})) = \# \left(\left(\{(q-1)\} \cup \bigcup_{m=1}^k \text{Sub}(\underline{n}^{(m)}) \right) / \sim \right).$$

(1) $\text{dep}(\underline{n}^{(k)}) = d_k \geq 2$.

(2) $\underline{n}^{(k)} \neq (n_1^{(k)}, n_1^{(k)}, \dots, n_1^{(k)})$.

(3) $\text{Sub}(\underline{n}^{(k)}) \setminus [\cup_{m=1}^{k-1} \text{Sub}(\underline{n}^{(m)})] = \{\underline{n}^{(k)}\}$.

(4) $\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{n}) | \underline{n} \in \cup_{m=1}^{k-1} \text{Sub}(\underline{n}^{(m)})) = \#((\{(q-1)\} \cup \cup_{m=1}^{k-1} \text{Sub}(\underline{n}^{(m)})) / \sim)$.

(5) *In the following, the (m, i, j) runs over all triples of integers such that*

$$1 \leq m \leq k-1, \quad 1 \leq j < i \leq d_m + 1, \quad i - j = d_k.$$

One of the following four conditions holds:

(5-1)

$$n_1^{(k)} \not\sim n_{i-1}^{(m)} \quad \text{or} \quad \underline{n}_{d_k+1,2}^{(k)} = (n_2^{(k)}, \dots, n_{d_k}^{(k)}) \not\sim \underline{n}_{i-1,j}^{(m)} = (n_j^{(m)}, \dots, n_{i-2}^{(m)})$$

and

$$n_1^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k+1,2}^{(k)} = (n_2^{(k)}, \dots, n_{d_k}^{(k)}) \not\sim \underline{n}_{i,j+1}^{(m)} = (n_{j+1}^{(m)}, \dots, n_{i-1}^{(m)})$$

for each (m, i, j) and $n_1^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim q-1$.

(5-1)'

$$n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)} = (n_1^{(k)}, \dots, n_{d_k-1}^{(k)}) \not\sim \underline{n}_{i-1,j}^{(m)} = (n_j^{(m)}, \dots, n_{i-2}^{(m)})$$

and

$$n_{d_k}^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)} = (n_1^{(k)}, \dots, n_{d_k-1}^{(k)}) \not\sim \underline{n}_{i,j+1}^{(m)} = (n_{j+1}^{(m)}, \dots, n_{i-1}^{(m)})$$

for each (m, i, j) and $n_{d_k}^{(k)}, \underline{n}_{d_k,1}^{(k)} \not\sim q-1$.

(5-2) *There exists exactly one triple (m_0, i_0, j_0) such that*

$$n_1^{(k)} \sim n_{i_0-1}^{(m_0)}, \quad n_{d_k}^{(k)} \sim n_{j_0}^{(m_0)}, \quad \underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{i_0,j_0+1}^{(m_0)}, \quad \underline{n}_{d_k+1,2}^{(k)} \sim \underline{n}_{i_0-1,j_0}^{(m_0)},$$

$n_\ell^{(k)} \neq n_{\ell+2}^{(k)}$ for some ℓ (resp. $\underline{n}^{(k)} \not\sim (n_2^{(m_0)}, n_1^{(m_0)})$) if $d_k \geq 3$ (resp. $d_k = 2$), and for other (m, i, j) 's

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i-1,j}^{(m)}$$

and

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i,j+1}^{(m)},$$

and $n_1^{(k)}, n_{d_k}^{(k)} \not\sim q-1$, and $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$.

(5-2)' There exists exactly one triple (m_0, i_0, j_0) such that

$$n_1^{(k)} \sim n_{j_0}^{(m_0)}, \quad n_{d_k}^{(k)} \sim n_{i_0-1}^{(m_0)}, \quad \underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{i_0-1,j_0}^{(m_0)}, \quad \underline{n}_{d_k+1,2}^{(k)} \sim \underline{n}_{i_0,j_0+1}^{(m_0)},$$

and for other (m, i, j) 's

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i-1,j}^{(m)}$$

and

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i,j+1}^{(m)},$$

and $n_1^{(k)}, n_{d_k}^{(k)} \not\sim q-1$, and $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$.

Proof of Theorem 1.2. We fix an order of the set $S = \{\underline{n}^{(1)}, \underline{n}^{(2)}, \dots\}$ such that $\text{dep}(\underline{n}^{(1)}) \leq \text{dep}(\underline{n}^{(2)}) \leq \dots$. For each $1 \leq k \leq \#S$, we show the equality

$$(1.1) \quad \text{tr.deg}_{\overline{K}}(\tilde{\pi}, \zeta(\underline{n}^{(1)}), \dots, \zeta(\underline{n}^{(k)})) = \#(\{(q-1), \underline{n}^{(1)}, \dots, \underline{n}^{(k)}\} / \sim)$$

by induction on k . If $\text{dep}(\underline{n}^{(k)}) = 1$, then the equality comes from the result of Chang and Yu ([5, Corollary 4.6]). Let $\text{dep}(\underline{n}^{(k)}) \geq 2$, then it is clear that the conditions (1), (2) and (3) of Theorem 1.5 hold. By the induction hypothesis, the condition (4) also holds. When $n_1^{(k)} \notin \{n_{j_1}, n_{j_2}\}$, the condition (5-1) holds if $d_k \geq 3$, and the condition (5-1), (5-2) or (5-2)' holds if $d_k = 2$. Similarly, when $n_{d_k}^{(k)} \notin \{n_{j_1}, n_{j_2}\}$, the condition (5-1)' holds if $d_k \geq 3$, and the condition (5-1)', (5-2) or (5-2)' holds if $d_k = 2$. When $n_1^{(k)} \sim n_{d_k}^{(k)}$ (this means that $n_1^{(k)} = n_{j_1}$ and $n_{d_k}^{(k)} = n_{j_2}$), then we have $d_k \geq 3$ by the definition of S , and the conditions (5-1) and (5-1)' hold. In any case, the condition (5) of Theorem 1.5 holds, and hence the equality (1.1) follows from Theorem 1.5. \square

Proof of Theorem 1.4. By Theorem 1.2, we may assume that $j_2 = j_1 + 1$. The proof is similar to that of Theorem 1.2. We fix an order on S as before, and show the equality (1.1) by induction. Let $\text{dep}(\underline{n}^{(k)}) \geq 2$. Then the conditions (1), (2) and (3) of Theorem 1.5 hold clearly, and the condition (4) follows from the induction hypothesis. In this case, the conditions (5-1) and (5-1)' hold. \square

The next proposition does not follow from Theorem 1.5, but we can show this by similar arguments of the proof of Theorem 1.5.

Proposition 1.6. *Let $\underline{n} = (n_1, n_2, n_3)$ be an index of depth three. If the n_i 's are "odd" and distinct from each other, then we have*

$$\text{tr.deg}_{\overline{K}}(\tilde{\pi}, \zeta(\underline{m}) | \underline{m} \in \text{Sub}(\underline{n})) = \#(\{(q-1)\} \cup \text{Sub}(\underline{n})) / \sim.$$

In Section 2, we define notations which are used in this paper and briefly review Papanikolas' theory of pre- t -motives. In Section 3 (resp. 4), we study “lifts” of p -th power (resp. Euler-Carlitz) relations. To apply Papanikolas' theory to MZVs which have p -th power or Euler-Carlitz relations, we need their lifts. In Section 5, we prove Theorem 1.5 and Proposition 1.6. The proofs are refinements of the proofs in [7].

§ 2. Preliminaries

We continue to use the notations of the Introduction. Let t be a new variable independent from θ . We fix an ∞ -adic valuation $|\cdot|_\infty$ on \mathbb{C}_∞ . Let $\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]] \mid f \text{ converges on } |t|_\infty \leq 1\}$ be the Tate algebra over \mathbb{C}_∞ and \mathbb{L} the fraction field of \mathbb{T} . For a formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ and an integer $n \in \mathbb{Z}$, we define the n -fold twisting of f by $f^{(n)} := \sum_i a_i^{q^n} t^i$. The fields $\overline{K}(t) \subset \mathbb{L}$ are stable under the action $f \mapsto f^{(n)}$ for each $n \in \mathbb{Z}$ and their fixed parts under the action $f \mapsto f^{(-1)}$ are $\mathbb{F}_q(t)$. Let $\|f\|_\infty := \max_i \{|a_i|_\infty\}$ denote the Gauss norm of f .

The formal power series

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{K}_\infty[[t]]$$

is an entire function and it is an element of \mathbb{T}^\times . Clearly, it satisfies

$$\Omega(\theta) = \frac{1}{\pi} \quad \text{and} \quad \Omega^{(-1)} = (t - \theta)\Omega.$$

Since $\Omega(t)$ has infinitely many zeros, it is transcendental over $\overline{K}(t)$.

Let $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$. For an integer $n \geq 0$ with q -adic expansion $n = \sum_i n_i q^i$ ($0 \leq n_i < q$), the factorial of Carlitz $\Gamma_{n+1} \in A$ is defined by

$$\Gamma_{n+1} := \prod_i D_i^{n_i}.$$

We set $D_n(t)$ (resp. $\Gamma_{n+1}(t)$) to be the inverse image of D_n (resp. Γ_{n+1}) by the \mathbb{F}_q -isomorphism $\mathbb{F}_q[t] \xrightarrow{\cong} A; t \mapsto \theta$. For an integer $n \geq 0$, the Bernoulli-Carlitz number $B_n \in A$ is defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma_{n+1}} z^n = z \left(\sum_{i=0}^{\infty} \frac{1}{D_i} z^{q^i} \right)^{-1}.$$

For each integer $n \geq 0$, Anderson and Thakur ([2, 3.7.1]) defined a polynomial $H_n \in A[t]$ by

$$\sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}(t)} z^n = \left(1 - \sum_{i=0}^{\infty} \frac{\prod_{j=1}^i (t^{q^i} - \theta^{q^j})}{D_i(t)} z^{q^i} \right)^{-1}.$$

We also set

$$S_i(n) := \sum_{\substack{a \in A: \text{monic} \\ \deg(a)=i}} \frac{1}{a^n}$$

for each $n \geq 1$ and $i \geq 0$. These satisfy

$$\|H_{n-1}\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}} \quad \text{and} \quad (H_{n-1}\Omega^n)^{(i)}(\theta) = \frac{\Gamma_n S_i(n)}{\tilde{\pi}^n}$$

for each $n \geq 1$ and $i \geq 0$ (see [2, 3.7.4], [3, 2.4.1]).

For an index $\underline{n} = (n_1, \dots, n_d)$, the formal power series

$$L_{\underline{n}}(t) := \sum_{i_1 > \dots > i_d \geq 0} \frac{H_{n_1-1}^{(i_1)} \cdots H_{n_d-1}^{(i_d)}}{((t - \theta^q) \cdots (t - \theta^{q^{i_1}}))^{n_1} \cdots ((t - \theta^q) \cdots (t - \theta^{q^{i_d}}))^{n_d}} \in \overline{K_\infty}[[t]],$$

converges on $|t|_\infty < |\theta|_\infty^q$ and it is an element of \mathbb{T} . Clearly, it satisfies

$$L_{\underline{n}}^{(-1)} = \frac{H_{n_d-1}^{(-1)}}{(t - \theta)^{n_1 + \dots + n_{d-1}}} L_{\underline{n}_{d1}} + \frac{L_{\underline{n}}}{(t - \theta)^{n_1 + \dots + n_d}},$$

where we set $L_{\underline{n}_{11}} = L_\emptyset := 1$ when $d = 1$. Anderson and Thakur ([3, 2.5.6]) showed that

$$L_{\underline{n}}(\theta) = \Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n}).$$

Next, we recall Papanikolas' theory of pre- t -motives. We do not give the complete details, but see [8] for more on this theory. See also [6, Section 2], [7, Section 3].

A *pre- t -motive* M is a finite dimensional $\overline{K}(t)$ -vector space equipped with a bijective additive map $\varphi: M \rightarrow M$ such that $\varphi(fm) = f^{(-1)}\varphi(m)$ for $f \in \overline{K}(t)$ and $m \in M$. We always assume that M is rigid analytically trivial. Thus such M is determined by the matrix $\Phi \in \text{GL}_r(\overline{K}(t))$ ($r := \dim M$) representing the φ -action with respect to a fixed basis, such that

$$\Psi^{(-1)} = \Phi\Psi$$

for some matrix $\Psi \in \text{GL}_r(\mathbb{L})$. The Betti realization $\omega(M)$ is defined and is functorial on M (see [8, 3.4 and 3.5]). The space $\omega(M)$ is an $\mathbb{F}_q(t)$ -vector space and its dimension over $\mathbb{F}_q(t)$ is equal to the dimension of M over $\overline{K}(t)$. The category of (rigid analytically trivial) pre- t -motives forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor ω . We denote by G_M the fundamental group of the Tannakian subcategory generated by M . When we fix a basis of M and choose a matrix Ψ as above, we also define

$$G_\Psi := \text{Spec}(\mathbb{F}_q(t)[X, 1/\det X]/\text{Ker } \nu) \subset \text{GL}_{r, \mathbb{F}_q(t)},$$

where $X = (X_{ij})$ is a matrix of $r \times r$ variables and ν is the $\mathbb{F}_q(t)$ -morphism defined by

$$\nu: \mathbb{F}_q(t)[X, 1/\det X] \rightarrow \mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}; \quad X_{ij} \mapsto \tilde{\Psi}_{ij}.$$

Here we set $\tilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$, where Ψ_1 (resp. Ψ_2) $\in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$ is the matrix defined by $(\Psi_1)_{ij} := \Psi_{ij} \otimes 1$ (resp. $(\Psi_2)_{ij} := 1 \otimes \Psi_{ij}$). Papanikolas ([8, Theorem 4.2.11]) showed that the scheme G_{Ψ} is a closed subgroup scheme of $\mathrm{GL}_{r, \mathbb{F}_q(t)}$. Moreover, he proved the following theorem:

Theorem 2.1 ([8, Theorems 4.3.1, 4.5.10, 5.2.2]). *There exists a natural isomorphism $G_{\Psi} \xrightarrow{\cong} G_M$ and the equality*

$$\dim G_{\Psi} = \mathrm{tr.deg}_{\overline{K}(t)} \overline{K}(t)(\Psi_{ij}|i, j)$$

holds. Moreover, this value is equal to

$$\mathrm{tr.deg}_{\overline{K}} \overline{K}(\Psi_{ij}(\theta)|i, j)$$

if $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$, $\det \Phi = c(t - \theta)^n$ for some $n \in \mathbb{Z}_{\geq 0}$ and $c \in \overline{K}^{\times}$, each entry of Ψ is entire and $\Psi \in \mathrm{GL}_r(\mathbb{T})$.

Remark. The last part of Theorem 2.1 is proved by using a very deep result in [1, Theorem 3.1.1], which is called the *ABP-criterion*. We use the last part of Theorem 2.1 to prove our theorems.

Example 2.2. Let $\underline{n} = (n_1, \dots, n_d)$ be an index. Let $M[\underline{n}]$ be the pre- t -motive defined by the $(d+1) \times (d+1)$ -matrix

$$\Phi[\underline{n}] := \begin{bmatrix} (t - \theta)^{n_1 + \dots + n_d} & 0 & 0 & \dots & 0 \\ H_{n_1-1}^{(-1)}(t - \theta)^{n_1 + \dots + n_d} & (t - \theta)^{n_2 + \dots + n_d} & 0 & \dots & 0 \\ 0 & H_{n_2-1}^{(-1)}(t - \theta)^{n_2 + \dots + n_d} & \ddots & & \vdots \\ \vdots & & \ddots & (t - \theta)^{n_d} & 0 \\ 0 & \dots & 0 & H_{n_d-1}^{(-1)}(t - \theta)^{n_d} & 1 \end{bmatrix}.$$

We also set

$$\Psi[\underline{n}] := \begin{bmatrix} \Omega^{n_1 + \dots + n_d} & 0 & 0 & \dots & 0 \\ \Omega^{n_1 + \dots + n_d} L_{\underline{n}_{21}} & \Omega^{n_2 + \dots + n_d} & 0 & \dots & 0 \\ \Omega^{n_1 + \dots + n_d} L_{\underline{n}_{31}} & \Omega^{n_2 + \dots + n_d} L_{\underline{n}_{32}} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \Omega^{n_d} & 0 \\ \Omega^{n_1 + \dots + n_d} L_{\underline{n}_{d+1,1}} & \Omega^{n_2 + \dots + n_d} L_{\underline{n}_{d+1,2}} & \dots & \Omega^{n_d} L_{\underline{n}_{d+1,d}} & 1 \end{bmatrix}.$$

Then we have $\Psi[\underline{n}]^{(-1)} = \Phi[\underline{n}]\Psi[\underline{n}]$. By Theorem 2.1, we have $G_{\Psi[\underline{n}]} \cong G_{M[\underline{n}]}$ and

$$\begin{aligned} \dim G_{\Psi[\underline{n}]} &= \mathrm{tr.deg}_{\overline{K}(t)} \overline{K}(t)(\Omega, L_{\underline{m}}|\underline{m} \in \mathrm{Sub}(\underline{n})) \\ &= \mathrm{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{m})|\underline{m} \in \mathrm{Sub}(\underline{n})). \end{aligned}$$

The matrix $\widetilde{\Psi}[\underline{n}] = (\widetilde{\Psi}[\underline{n}]_{ij})$ is calculated as

$$\widetilde{\Psi}[\underline{n}]_{ij} = (\Omega^{-1} \otimes \Omega)^{n_i + \dots + n_d} \sum_{s=j}^i \sum_{r=0}^{i-s} (-1)^r \sum_{\substack{s=i_0 < i_1 < \dots \\ < i_{r-1} < i_r = i}} L_{i_1 i_0} \cdots L_{i_r i_{r-1}} \otimes \Omega^{n_j + \dots + n_{i-1}} L_{s j}$$

for each $j < i$, where we denote $L_{k\ell} := L_{\underline{n}_{k\ell}}$.

Example 2.3. For $k = 1, 2$, let $\Phi_k \in \mathrm{GL}_{r_k}(\overline{K}(t))$ and $\Psi_k \in \mathrm{GL}_{r_k}(\mathbb{L})$ be matrices such that $\Psi_k^{(-1)} = \Phi_k \Psi_k$, and let M_k be the pre- t -motive defined by Φ_k . Since M_k is a direct factor of $M_1 \oplus M_2$, there exists a surjective map

$$G_{\Psi_1 \oplus \Psi_2} \cong G_{M_1 \oplus M_2} \twoheadrightarrow G_{M_k} \cong G_{\Psi_k}$$

by Tannakian duality and Theorem 2.1. This coincides with the restriction of the k -th projection $\mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \twoheadrightarrow \mathrm{GL}_{r_k}$.

§ 3. Lifts of p -th power relations

In this section, we study p -th power relations among $L_{\underline{n}}$'s, which are lifts of p -th power relations among MZVs. For an index $\underline{n} = (n_1, \dots, n_d)$, we use the notations

$$\Omega^{\underline{n}} := \Omega^{n_1 + \dots + n_d} \quad \text{and} \quad \underline{n}' := (n_1, \dots, n_{d-1}).$$

Lemma 3.1. *For each positive integer $n \geq 1$ and each non-negative integer $e \geq 0$, we have*

$$\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)} = \left(\frac{H_{n-1}}{\Gamma_n(t)} \right)^{p^e}.$$

Proof. We have

$$\left(\frac{H_{s-1}}{\Gamma_s(t)} (\theta^{q^{-i}}) \right)^{q^i} = \left(\frac{H_{s-1}}{\Gamma_s(t)} \right)^{(i)} (\theta) = \frac{S_i(s)}{\widetilde{\pi}^s (\Omega^s)^{(i)}(\theta)}.$$

Therefore

$$\begin{aligned} \left(\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)} (\theta^{q^{-i}}) \right)^{q^i} &= \frac{S_i(p^e n)}{\widetilde{\pi}^{p^e n} (\Omega^{p^e n})^{(i)}(\theta)} = \left(\frac{S_i(n)}{\widetilde{\pi}^n (\Omega^n)^{(i)}(\theta)} \right)^{p^e} \\ &= \left(\left(\frac{H_{n-1}}{\Gamma_n(t)} (\theta^{q^{-i}}) \right)^{q^i} \right)^{p^e} = \left(\left(\frac{H_{n-1}}{\Gamma_n(t)} \right)^{p^e} (\theta^{q^{-i}}) \right)^{q^i} \end{aligned}$$

for each $i \geq 0$. Thus we have

$$\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)} (\theta^{q^{-i}}) = \left(\frac{H_{n-1}}{\Gamma_n(t)} \right)^{p^e} (\theta^{q^{-i}})$$

for each $i \geq 0$. □

We set

$$\gamma_{e,n} := \frac{H_{p^e n-1}}{H_{n-1}^{p^e}} = \frac{\Gamma_{p^e n}(t)}{\Gamma_n(t)^{p^e}} \in \mathbb{F}_q(t)^\times$$

and

$$\gamma_{e,\underline{n}} := \prod_{i=1}^d \gamma_{e,n_i}$$

for any index $\underline{n} = (n_1, \dots, n_d)$. When $p^{-e}\underline{n} \in \mathbb{Z}^d$, we also set

$$\gamma_{-e,\underline{n}} := \gamma_{e,p^{-e}\underline{n}}^{-p^{-e}}.$$

The next lemma gives a p -th power relation among $L_{\underline{n}}$ and $L_{p^e \underline{n}}$, which is a lift of the p -th power relation $\zeta(p^e \underline{n}) = \zeta(\underline{n})^{p^e}$.

Lemma 3.2. *For each index \underline{n} of depth d and each integer $e \in \mathbb{Z}$ such that $p^e \underline{n} \in \mathbb{Z}^d$, we have*

$$L_{p^e \underline{n}} = \gamma_{e,\underline{n}} L_{\underline{n}}^{p^e}.$$

Proof. By the definition of $\gamma_{e,\underline{n}}$ for negative integers e , we may assume that $e \geq 0$. We prove this equality by induction on d . When $d = 0$, it is clear. We take $d \geq 1$ and assume that the above equality holds for any indices whose depths are lower than d . Then we have

$$\begin{aligned} & (\Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e})^{(-1)} \\ &= H_{p^e n_d-1}^{(-1)} (t - \theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}'} + \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} \left(H_{n_d-1}^{(-1)} (t - \theta)^{n_d} \Omega^{\underline{n}} L_{\underline{n}'} + \Omega^{\underline{n}} L_{\underline{n}} \right)^{p^e} \\ &= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t - \theta)^{p^e n_d} \Omega^{p^e \underline{n}} \left(H_{p^e n_d-1}^{(-1)} L_{p^e \underline{n}'} - \gamma_{e,\underline{n}'} \gamma_{e,n_d} (H_{n_d-1}^{(-1)})^{p^e} L_{\underline{n}'}^{p^e} \right) \\ &= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t - \theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}'} \left(H_{p^e n_d-1}^{(-1)} - \gamma_{e,n_d} (H_{n_d-1}^{(-1)})^{p^e} \right) \\ &= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t - \theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}'} \left(H_{p^e n_d-1}^{(-1)} - \gamma_{e,n_d}^{(-1)} (H_{n_d-1}^{(-1)})^{p^e} \right) \\ &= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e}. \end{aligned}$$

Thus

$$\Omega^{p^e \underline{n}} L_{p^e \underline{n}} = \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + c$$

for some $c \in \mathbb{F}_q(t)$. Since we have

$$\begin{aligned} (\gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e})(\theta) &= \frac{\Gamma_{p^e n_1}}{\Gamma_{n_1}^{p^e}} \cdots \frac{\Gamma_{p^e n_d}}{\Gamma_{n_d}^{p^e}} \left(\frac{\Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n})}{\widetilde{\pi}^{n_1 + \cdots + n_d}} \right)^{p^e} \\ &= \frac{\Gamma_{p^e n_1} \cdots \Gamma_{p^e n_d} \zeta(p^e \underline{n})}{\widetilde{\pi}^{p^e n_1 + \cdots + p^e n_d}} \\ &= (\Omega^{p^e \underline{n}} L_{p^e \underline{n}})(\theta), \end{aligned}$$

we conclude $c(\theta) = 0$, and hence $c = 0$. \square

Lemma 3.3. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index and $\Psi := \Psi[\underline{n}] \in \mathrm{GL}_{d+1}(\mathbb{L})$ the matrix defined in Example 2.2. Take $1 \leq j < i \leq d+1$ and $1 \leq \ell < k \leq d+1$ such that $\underline{n}_{ij} = p^e \underline{n}_{k\ell}$ for some integer $e \in \mathbb{Z}$. Then the equality*

$$\tilde{\Psi}_{ij}/\tilde{\Psi}_{ii} = \gamma_{e, \underline{n}_{k\ell}} (\tilde{\Psi}_{k\ell}/\tilde{\Psi}_{kk})^{p^e}$$

holds.

Proof. We may assume that $e \geq 0$. By Example 2.2, we have

$$\begin{aligned} \tilde{\Psi}_{ij}/\tilde{\Psi}_{ii} &= \sum_{s=j}^i \sum_{r=0}^{i-s} (-1)^r \sum_{\substack{s=i_0 < i_1 < \dots \\ < i_{r-1} < i_r = i}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{ij}} L_{\underline{n}_{sj}} \\ &= \sum_{s=\ell}^k \sum_{r=0}^{k-s} (-1)^r \sum_{\substack{s=i_0 < i_1 < \dots \\ < i_{r-1} < i_r = k}} L_{p^e \underline{n}_{i_1 i_0}} \cdots L_{p^e \underline{n}_{i_r i_{r-1}}} \otimes \Omega^{p^e \underline{n}_{k\ell}} L_{p^e \underline{n}_{s\ell}} \\ &= \sum_{s=\ell}^k \sum_{r=0}^{k-s} (-1)^r \sum_{\substack{s=i_0 < i_1 < \dots \\ < i_{r-1} < i_r = k}} \gamma_{e, \underline{n}_{i_1 i_0}} \cdots \gamma_{e, \underline{n}_{i_r i_{r-1}}} \gamma_{e, \underline{n}_{s\ell}} (L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}})^{p^e} \\ &= \gamma_{e, \underline{n}_{k\ell}} \left(\sum_{s=\ell}^k \sum_{r=0}^{k-s} (-1)^r \sum_{\substack{s=i_0 < i_1 < \dots \\ < i_{r-1} < i_r = k}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}} \right)^{p^e} \\ &= \gamma_{e, \underline{n}_{k\ell}} (\tilde{\Psi}_{k\ell}/\tilde{\Psi}_{kk})^{p^e}. \end{aligned}$$

\square

§ 4. Lifts of Euler-Carlitz relations

In this section, we study Euler-Carlitz relations among Ω and L_n 's, which are lifts of Euler-Carlitz relations among $\tilde{\pi}$ and Carlitz zeta values at positive “even” integers. Let $n \geq 1$ be a positive “even” integer. By [6, Remark 3.3], there exist $c_n \in \mathbb{F}_q(t)^\times$ and $f_n \in \overline{K}(t)$ such that

$$\Omega^n L_n - c_n = f_n \Omega^n.$$

This gives a lift of the Euler-Carlitz relation at n . The c_n is determined by

$$c_n(\theta) = \frac{\Gamma_n \zeta(n)}{\tilde{\pi}^n} = \frac{\Gamma_n B_n}{\Gamma_{n+1}} \in K^\times.$$

Let $\underline{n} = (n_1, \dots, n_d)$ be an index and take j such that n_j is “even”. Then we have

$$\tilde{\Psi}_{j+1,j}/\tilde{\Psi}_{j+1,j+1} = c_{n_j}(1 - (\Omega^{-1} \otimes \Omega)^{n_j}),$$

where $\Psi := \Psi[\underline{n}]$ is the matrix defined in Example 2.2.

§ 5. Proofs

In this section, we always assume that algebraic groups are defined over $\mathbb{F}_q(t)$. We need the following lemma. This can be proved easily and we omit the proof.

Lemma 5.1. *Let $V \subset \mathbb{G}_a^r$ be an algebraic subgroup of dimension zero. Let $m_1, \dots, m_r \in \mathbb{Z}$ be non-zero integers. Assume that V is stable under the \mathbb{G}_m -action on \mathbb{G}_a^r defined by*

$$a.(x_1, \dots, x_r) = (a^{m_1}x_1, \dots, a^{m_r}x_r) \quad (a \in \mathbb{G}_m, (x_i) \in \mathbb{G}_a^r).$$

Then $V(\overline{\mathbb{F}_q(t)})$ is trivial.

From now on, we identify group schemes over $\mathbb{F}_q(t)$ with the sets of $\overline{\mathbb{F}_q(t)}$ -valued points of them. We use letters $a, x_{ij}^{(m)}, \dots$ for coordinate variables of algebraic groups.

Proof of Theorem 1.5. For $1 \leq \ell \leq k$, let $G^{\leq \ell}$ be the algebraic group defined by the matrix $[\Omega] \oplus \bigoplus_{m=1}^{\ell} \Psi[\underline{n}_m]$. Then $G^{\leq \ell}$ is an algebraic subgroup of

$$\left\{ \begin{aligned} & \left[a \right] \oplus \bigoplus_{m=1}^{\ell} \begin{bmatrix} a^{n_1^{(m)} + \dots + n_{d_m}^{(m)}} & & \\ & x_{21}^{(m)} & \ddots \\ & \vdots & \ddots & a^{n_{d_m}^{(m)}} \\ x_{d_m+1,1}^{(m)} & \dots & x_{d_m+1,d_m}^{(m)} & 1 \end{bmatrix} \in \mathbb{G}_m \times \prod_{m=1}^{\ell} \mathrm{GL}_{d_m+1} \\ & \left| \begin{aligned} & x_{i'j'}^{(m')} / a^{n_{i'}^{(m')} + \dots + n_{d_{m'}}^{(m')}} = \gamma_{e, \underline{n}_{ij}^{(m)}} \left(x_{ij}^{(m)} / a^{n_i^{(m)} + \dots + n_{d_m}^{(m)}} \right)^{p^e} \\ & \quad \text{if } \underline{n}_{i'j'}^{(m')} = p^e \underline{n}_{ij}^{(m)}, \\ & x_{j+1,j}^{(m)} / a^{n_{j+1}^{(m)} + \dots + n_{d_m}^{(m)}} = c_{n_j^{(m)}} (1 - a^{n_j^{(m)}}) \\ & \quad \text{if } n_j^{(m)} \text{ is “even”} \end{aligned} \right. \end{aligned} \right\}.$$

By Theorem 2.1 and the condition (4), this inclusion is actually an equality for $1 \leq \ell \leq k-1$. It suffices to show that this inclusion is actually an equality for $\ell = k$. We already have

$$\dim G^{\leq k-1} \leq \dim G^{\leq k} \leq \dim G^{\leq k-1} + 1$$

by the condition (3) and it suffices to show that the second inequality is an equality. Let

$$\psi: G^{\leq k} \twoheadrightarrow G^{\leq k-1} \quad \text{and} \quad \pi^{\leq \ell}: G^{\leq \ell} \twoheadrightarrow G_{\Omega} \cong \mathbb{G}_m \quad (1 \leq \ell \leq k)$$

be the surjections obtained as in Example 2.3. We set $V^{\leq \ell} := \text{Ker } \pi^{\leq \ell}$ to be the unipotent radical of $G^{\leq \ell}$. Then we have the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & V^{\leq k} & \longrightarrow & G^{\leq k} & \xrightarrow{\pi^{\leq k}} & \mathbb{G}_m \longrightarrow 1 \\ & & \downarrow \psi' & & \downarrow \psi & & \parallel \\ 1 & \longrightarrow & V^{\leq k-1} & \longrightarrow & G^{\leq k-1} & \xrightarrow{\pi^{\leq k-1}} & \mathbb{G}_m \longrightarrow 1 \end{array}$$

with exact rows, where ψ' is the restriction of ψ to $V^{\leq k}$. The morphism ψ' is surjective. It is clear that $\text{Ker } \psi'$ is a normal subgroup of $G^{\leq k}$. The conjugate action $X \mapsto A^{-1}XA$ of $G^{\leq k}$ on $\text{Ker } \psi'$ factors through the action of \mathbb{G}_m on $\text{Ker } \psi'$ by

$$a.(\dots, 0, x_{d_k+1,1}^{(k)}, 0, \dots) = (\dots, 0, a^{n_1^{(k)} + \dots + n_{d_k}^{(k)}} x_{d_k+1,1}^{(k)}, 0, \dots).$$

Now, we assume that $\dim G^{\leq k} = \dim G^{\leq k-1}$ and we shall induce a contradiction. Since $\dim \text{Ker } \psi' = 0$, the group $\text{Ker } \psi'$ is trivial by Lemma 5.1. Let $X = (x_{ij}^{(m)})$ (resp. $A = (a_{ij}^{(m)})$) be any element of $V^{\leq k}$ such that $x_{ij}^{(m)} = 0$ if $i - j \neq 1$ (resp. $a_{ij}^{(m)} = 0$ if $i - j \neq d_k - 1$) and $(m, i, j) \neq (k, d_k + 1, 1)$. For each $1 \leq m \leq k$, we set $x_j^{(m)} := x_{j+1,j}^{(m)}$ ($1 \leq j \leq d_m$) and $a_j^{(m)} := a_{d_k-1+j,j}^{(m)}$ ($1 \leq j \leq d_m - d_k + 2$). We also set $d := d_k - 1$ (which is ≥ 1 by the condition (1)). Then the m -th component of $A^{-1}XA$ is equal to

$$\begin{bmatrix} 1 & & & & & & & & & & \\ x_1^{(m)} & & & & & & & & & & \\ 0 & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ 0 & & & & & & & & & & \\ a_1^{(m)} x_{d+1}^{(m)} - a_2^{(m)} x_1^{(m)} & & & & & & & & & & \\ 0 & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ 0 & & & & & & & & & & \\ -a_{d+2}^{(m)} (a_1^{(m)} x_{d+1}^{(m)} - a_2^{(m)} x_1^{(m)}) & & & & & & & & & & \\ 0 & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ 0 & & & & & & & & & & \\ a_{d+2}^{(m)} a_{2d+2}^{(m)} (a_1^{(m)} x_{d+1}^{(m)} - a_2^{(m)} x_1^{(m)}) & & & & & & & & & & \\ 0 & & & & & & & & & & \\ \vdots & & & & & & & & & & \end{bmatrix}$$

for each $1 \leq m < k$. Thus the (i, j) -th entry of the m -th (resp. k -th) component of $A^{-1}XA$ is

$$\begin{cases} 1 & (i - j = 0) \\ x_j^{(m)} & (i - j = 1) \\ (a_j^{(m)} x_{d+j}^{(m)} - a_{j+1}^{(m)} x_j^{(m)}) (-1)^{r-1} \prod_{s=1}^{r-1} a_{sd+j+1}^{(m)} & (i - j = rd + 1, r \geq 1) \\ \left(\text{resp. } a_1^{(k)} x_{d+1}^{(k)} - a_2^{(k)} x_1^{(k)} + x_{d_k+1,1}^{(k)} \right) & (\text{resp. } i - j = d + 1 = d_k) \\ 0 & (\text{otherwise}) \end{cases}$$

Therefore, if the equalities

$$(5.1) \quad a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} - a_{i,j+1}^{(m)} x_{j+1,j}^{(m)} = 0 \quad (1 \leq m < k, \quad 1 \leq j < i \leq d_m + 1, \quad i - j = d_k)$$

hold, then $X^{-1}A^{-1}XA \in \text{Ker } \psi' = \{1\}$ and the equality

$$(5.2) \quad a_{d_k,1}^{(k)} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$$

must hold. We show that this implication induces a contradiction and hence we have $\dim G^{\leq k} = \dim G^{\leq k-1} + 1$.

First, we assume that the condition (5-1) holds. If $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$, we can take

$$a_{d_k+1,2}^{(k)} x_{21}^{(k)} \neq 0, \quad a_{d_k,1}^{(k)} x_{d_k+1,d_k}^{(k)} = a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each (m, i, j) . Then the equalities (5.1) hold and hence the equality (5.2) also holds. However, it becomes $a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$. This is a contradiction. If $\underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{d_k+1,2}^{(k)}$ (and hence $n_1^{(k)} \sim n_{d_k}^{(k)}$), there exists an integer e such that $\underline{n}_{d_k,1}^{(k)} = p^e \underline{n}_{d_k+1,2}^{(k)}$. Then we can take

$$a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each (m, i, j) and the equality (5.2) becomes

$$\gamma_{e, \underline{n}_{d_k+1,2}^{(k)}} (a_{d_k+1,2}^{(k)})^{p^e} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$$

for each $a_{d_k+1,2}^{(k)}$ and $x_{21}^{(k)}$. Then e must be zero. However, since $\underline{n}^{(k)} \neq (n_1^{(k)}, \dots, n_1^{(k)})$ by the condition (2), e is non-zero. This is a contradiction.

Similarly, when the condition (5-1)' holds, we obtain a contradiction.

Next, we assume that the condition (5-2) holds. Then we can take

$$x_{21}^{(k)}, x_{d_k+1,d_k}^{(k)} \neq 0, \quad a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each $(m, i, j) \neq (m_0, i_0, j_0)$ and $a_{d_k,1}^{(k)}, a_{d_k+1,2}^{(k)}$ as any elements. There exist integers $e_1, e_2, e_3, e_4 \in \mathbb{Z}$ such that

$$\underline{n}_{i_0-1,j_0}^{(m_0)} = p^{e_1} \underline{n}_{d_k+1,2}^{(k)}, \quad \underline{n}_{i_0,j_0+1}^{(m_0)} = p^{e_2} \underline{n}_{d_k,1}^{(k)}, \quad n_{j_0}^{(m_0)} = p^{e_3} n_{d_k}^{(k)}, \quad n_{i_0-1}^{m_0} = p^{e_4} n_1^{(k)}.$$

Then the (m_0, i_0, j_0) -th equality of the equalities (5.1) becomes

$$\gamma_{e_1, \underline{n}_{d_k+1,2}}^{(k)} (a_{d_k+1,2}^{(k)})^{p^{e_1}} \gamma_{e_4, n_1}^{(k)} (x_{21}^{(k)})^{p^{e_4}} - \gamma_{e_2, \underline{n}_{d_k,1}}^{(k)} (a_{d_k,1}^{(k)})^{p^{e_2}} \gamma_{e_3, n_{d_k}}^{(k)} (x_{d_k+1,d_k}^{(k)})^{p^{e_3}} = 0.$$

We take any $a_{d_k+1,2}^{(k)}$ and set

$$a_{d_k,1}^{(k)} := \left(\frac{\gamma_{e_1, \underline{n}_{d_k+1,2}}^{(k)} \gamma_{e_4, n_1}^{(k)} (x_{21}^{(k)})^{p^{e_4}}}{\gamma_{e_2, \underline{n}_{d_k,1}}^{(k)} \gamma_{e_3, n_{d_k}}^{(k)} (x_{d_k+1,d_k}^{(k)})^{p^{e_3}}} \right)^{p^{-e_2}} (a_{d_k+1,2}^{(k)})^{p^{e_1-e_2}}.$$

Then the equalities (5.1) hold and the equality (5.2) becomes

$$\left(\frac{\gamma_{e_1, \underline{n}_{d_k+1,2}}^{(k)} \gamma_{e_4, n_1}^{(k)} (x_{21}^{(k)})^{p^{e_4}}}{\gamma_{e_2, \underline{n}_{d_k,1}}^{(k)} \gamma_{e_3, n_{d_k}}^{(k)} (x_{d_k+1,d_k}^{(k)})^{p^{e_3}}} \right)^{p^{-e_2}} (a_{d_k+1,2}^{(k)})^{p^{e_1-e_2}} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$$

and holds for each $a_{d_k+1,2}^{(k)}$. This implies $e_1 = e_2$. Then we have $n_\ell^{(k)} = n_{\ell+2}^{(k)}$ for each ℓ (resp. $\underline{n}^{(k)} \sim (n_2^{(m_0)}, n_1^{(m_0)})$) if $d_k \geq 3$ (resp. $d_k = 2$). In any case, we obtain a contradiction.

Similarly, when the condition (5-2)' holds, we have $\underline{n}^{(k)} = (n_1^{(k)}, \dots, n_1^{(k)})$ (resp. $\underline{n}^{(k)} \sim \underline{n}^{(m_0)}$) if $d_k \geq 3$ (resp. $d_k = 2$). In any case, we obtain a contradiction. \square

Proof of Proposition 1.6. We use the notations in the proof of Theorems 1.5. We set

$$\begin{aligned} \underline{n}^{(1)} &:= n_1, \quad \underline{n}^{(2)} := n_2, \quad \underline{n}^{(3)} := n_3, \\ \underline{n}^{(4)} &:= (n_1, n_2), \quad \underline{n}^{(5)} := (n_2, n_3), \quad \underline{n}^{(6)} := (n_1, n_2, n_3). \end{aligned}$$

By Theorem 1.5 and the result of Chang and Yu ([5, Corollary 4.6]), we have

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{n}^{(m)})) | 1 \leq m \leq 4 = \#(\{ (q-1) \} \cup \{ \underline{n}^{(m)} | 1 \leq m \leq 4 \}) / \sim.$$

If we prove

$$\text{tr.deg}_{\overline{K}} \overline{K}(\tilde{\pi}, \zeta(\underline{n}^{(m)})) | 1 \leq m \leq 5 = \#(\{ (q-1) \} \cup \{ \underline{n}^{(m)} | 1 \leq m \leq 5 \}) / \sim,$$

then Proposition 1.6 follows from Theorem 1.5.

Assume that $\underline{n}^{(5)} \not\sim \underline{n}^{(4)}$ and $\dim G^{\leq 5} = \dim G^{\leq 4}$. In this case, the equality $a_{21}^{(4)} x_{32}^{(4)} - a_{32}^{(4)} x_{21}^{(4)} = 0$ implies the equality $a_{21}^{(5)} x_{32}^{(5)} - a_{32}^{(5)} x_{21}^{(5)} = 0$. We may assume that $n_1 \sim n_2 \sim n_3$, otherwise we obtain a contradiction from [7, Theorem 1.1] and Theorem 1.4. We set

$$\begin{aligned} a_1 &:= a_{21}^{(4)}, \quad a_2 := a_{32}^{(4)} = a_{21}^{(5)}, \quad a_3 := a_{32}^{(5)}, \\ x_1 &:= x_{21}^{(4)}, \quad x_2 := x_{32}^{(4)} = x_{21}^{(5)}, \quad x_3 := x_{32}^{(5)}. \end{aligned}$$

For each j , we have $n_j = p^{e_j}n$ for some $n \geq 1$ and $e_j \geq 0$ with $\min\{e_j\} = 0$. We set $a := a_{j_0}$ and $x := x_{j_0}$ for some j_0 such that $e_{j_0} = 0$. Thus we have $a_j = \gamma_{e_j,n}a^{p^{e_j}}$ and $x_j = \gamma_{e_j,n}x^{p^{e_j}}$ for each j . Then

$$a^{p^{e_1}}x^{p^{e_2}} - a^{p^{e_2}}x^{p^{e_1}} = 0 \text{ implies } a^{p^{e_2}}x^{p^{e_3}} - a^{p^{e_3}}x^{p^{e_2}} = 0$$

for any $a, x \in \overline{\mathbb{F}_q(t)}$. Since $e_1 \neq e_2$, we conclude that $e_1 - e_2$ divides $e_2 - e_3$. By symmetric arguments, since $e_2 \neq e_3$, we conclude that $e_3 - e_2$ divides $e_2 - e_1$. This means that $e_1 - e_2 = \pm(e_2 - e_3)$. However this is a contradiction because we assume that $\underline{n}^{(5)} \not\sim \underline{n}^{(4)}$ and $e_1 \neq e_3$. \square

Acknowledgments. The author would like to thank Yuichiro Taguchi who carefully read a preliminary version of this paper and gave him many comments. This work was partially supported by the JSPS Research Fellowships for Young Scientists.

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